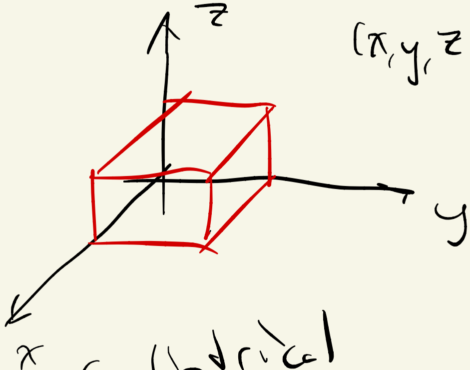
8-10-21

Last time... Triple integrals

3 coord systems:

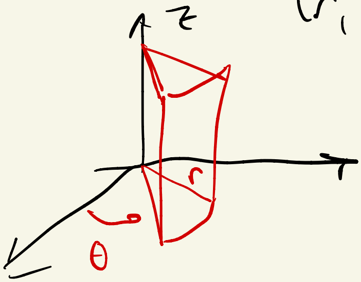
· Rectangular

(x, y, z)



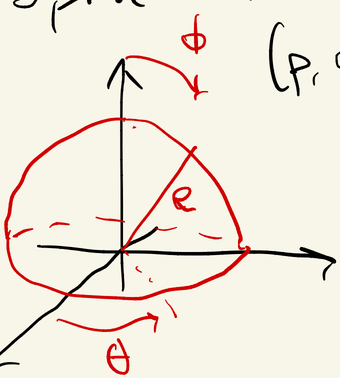
Cylindrical

(r, θ, z)



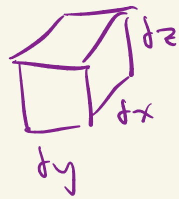
Spherical

(ρ, ϕ, θ)



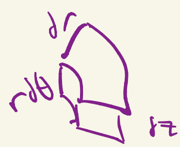
Volume elements:

Rectangular prisms



$dV = dx\,dy\,dz$

Flat wedges



$dV = r\,d\theta\,dr\,dz$

Sphere wedges

$dV = \rho\,d\phi\,\rho\sin\phi\,d\theta\,d\rho$

$= \rho^2 \sin\phi\,d\rho\,d\phi\,d\theta$

15.8

- Moving between coordinate systems, how do we track how dV changes (and bounds for integrating)

Want: give me a coordinate transformation T. How does T cause change in dV?

Recall: 21B

u-substitution — how to undo the chain rule?

$$\int_a^b f(x)\,dx \overset{?}{=} \int_{T^{-1}(a)}^{T^{-1}(b)} f(T(u))\,T'(u)\,du$$

chain rule

$$x = T(u)$$

$$\rightarrow dx = T'(u)\,du$$

$$F(T(u))' = F'(T(u))T'(u)$$
$$= f(T(u))T'(u)$$

If $F' = f$, then

$$\int_{T^{-1}(a)}^{T^{-1}(b)} f(T(u))\,T'(u)\,du = \int_{T^{-1}(a)}^{T^{-1}(b)} (F \circ T)'(u)$$

Fundamental thm calc

$$= \Big[(F \circ T)(u)\Big]_{T^{-1}(a)}^{T^{-1}(b)}$$

$T \circ T^{-1} = \text{identity}$

$$= (F \circ T)(T^{-1}(b)) - (F \circ T)(T^{-1}(a))$$

$$= F(b) - F(a)$$

$$\int_a^b f(x)\,dx = \int_{T^{-1}(a)}^{T^{-1}(b)} f(T(u))\,T'(u)\,du$$

$$x = T(u)$$
$$dx = T'(u)\,du$$

Ex:

$$\int_0^{\sqrt{\pi}} x\sin(x^2)\,dx = \int_0^{\pi} \tfrac{1}{2}\sin(u)\,du$$

$$u = x^2 \qquad du = 2x\,dx \quad \Rightarrow \quad x = T(u) = \sqrt{u}, \quad dx = \frac{1}{2x}\,du$$

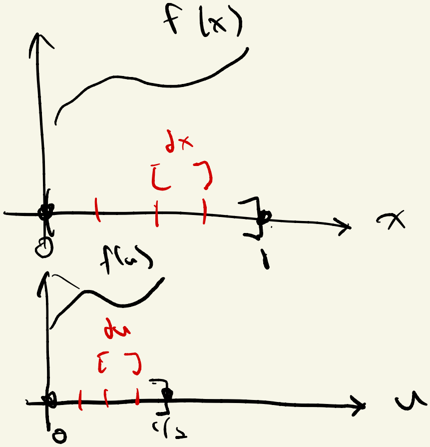
$$T^{-1}(x) = x^2$$
$$\Rightarrow \quad T^{-1}(0) = 0$$
$$T^{-1}(\sqrt{\pi}) = \pi$$

Why did this work?

It's all about the chain rule:

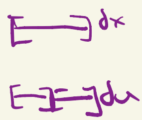
1D:

really more Δx ... but that's ok



$x = T(u)$, we'll choose

$$x = 2u$$
$$\Rightarrow dx = 2\,du$$



"Each change in u is half the size of each change in x"

This works for <u>any</u> differentiable transformation T.

The important part is compensating for change

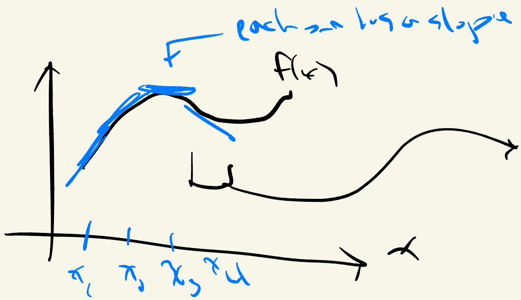
$$dx = T'(u)\,du$$

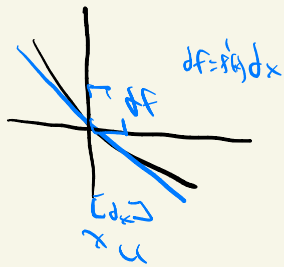
remind you of

$$dx\,dy = r\,dr\,d\theta$$

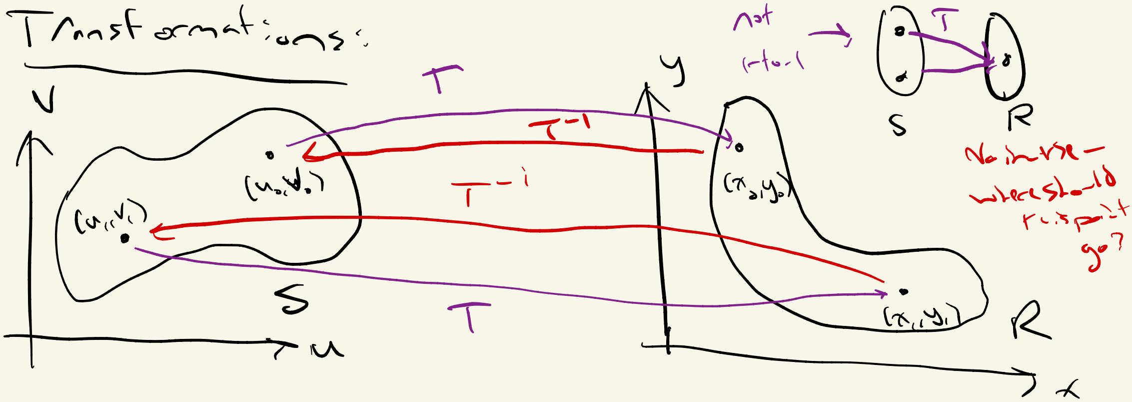
"something like a derivative" from a chain rule

Note: this only works because differentiable functions are "locally linearly approximable": Tangent line approx works everywhere





Transformations:



T is a <u>transformation</u>, $T : S \to R$

when T connects every point in $T(S)$ to <u>exactly one</u> point in S, T is "one-to-one".

If T is one-to-one, it has an inverse T^{-1}:

- $T(S)$ "Image". In this class, you can assume that $T(S) = R$ (onto)
- $T^{-1}(R)$ "Preimage". Again, $T^{-1}(R) = S$.

  We will only work with 1-to-1 transformation

Ex: Coord transformations

$$T(x,y,z) = (r,\theta,\phi)$$

$$(x,y,z) = T^{-1}(r,\theta,\phi)$$

we know <u>exactly</u> which (x,y,z) correspond to which (r,θ,ϕ). (1-to-1)

- $T(u,v) = (x,y)$
- $T^{-1}(x,y) = (u,v)$

Ex:

$S =$ square, $\quad 0 \le u, v \le 1$

what is $T(S)$, where T is

$$x = u + v$$
$$y = 2v$$

Fact: T is a linear transformation. we can write it as a matrix:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$

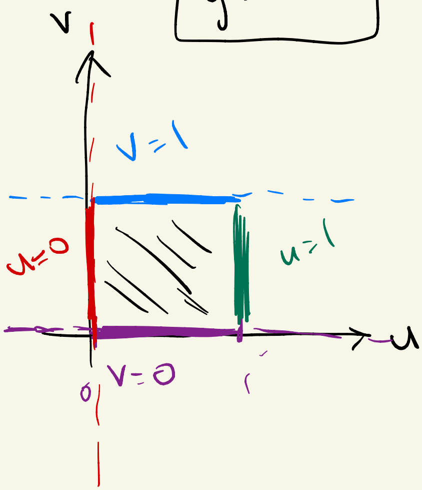
$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \cdot u + 1v \\ 0 \cdot u + 2v \end{bmatrix} = \begin{bmatrix} u + v \\ 2v \end{bmatrix}$$

Soln:

Linear transformations (like T) map lines to lines

$$\boxed{\begin{aligned} x &= u + v \\ y &= 2v \end{aligned}}$$

*Warning: Not all linear transformations have matrices. But all of ours do in this class



- $v = 0 \xmapsto{T} y = 0$

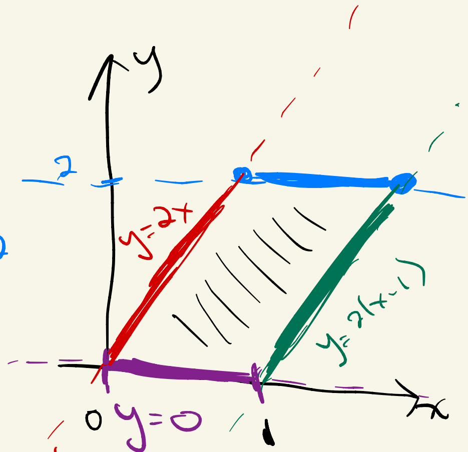
  $y = 0, \ x = u$

  $\Rightarrow x = 0$ to $x = 1$

- $v = 1 \xmapsto{T} y = 2$

  $y = 2, \ x = u + 1$

  $0 \le u \le 1$



- $u = 0$ from $v = 0$ to $v = 1$

  $x = v$

  $y = 2v \Rightarrow y = 2x$

- $u = 1$ from $v = 0$ to $v = 1$

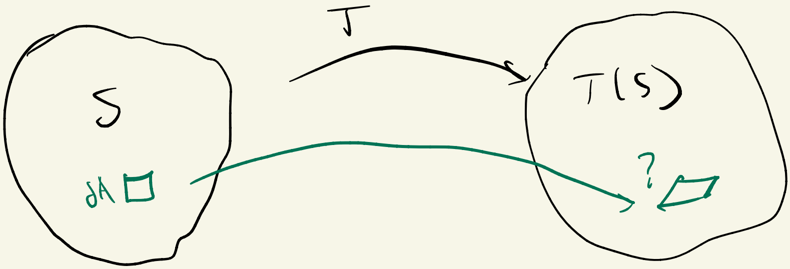
  $x = 1 + v \Rightarrow v = x - 1$

  $\Rightarrow y = 2(x - 1)$

11:02

What happens to area differential?

- Differentiable maps T have linear approximations

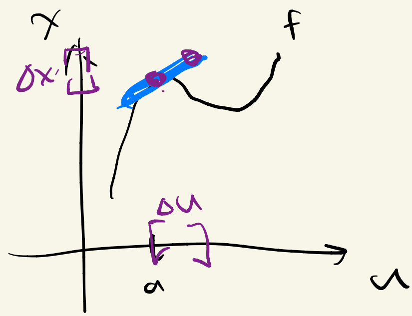


Recall polar:

$dA = dy\,dx$ $\quad\longrightarrow\quad$ $dA = r\,d\theta\,dr$

What's happening?

Recall dA: linear approximation $f: \mathbb{R} \to \mathbb{R}$



- Nearby $u=a$...

can linearize

- slope $= f'(a)$

Δu

$\Delta x = f'(a)\,\Delta u$

$\Rightarrow dx = f'(a)\,du \leftarrow \dfrac{dx}{du} = f'(a)$

If we want this to work at any u,

$dx = f'(u)\,du$

What about 2D?

Derivatives encoded by a matrix: Jacobian

$(x,y) = T(u,v)$

$$J(u,v) = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix}$$

* warning: book calls det(Jacobian) The Jacobian. When I say Jacobian, I mean this matrix.

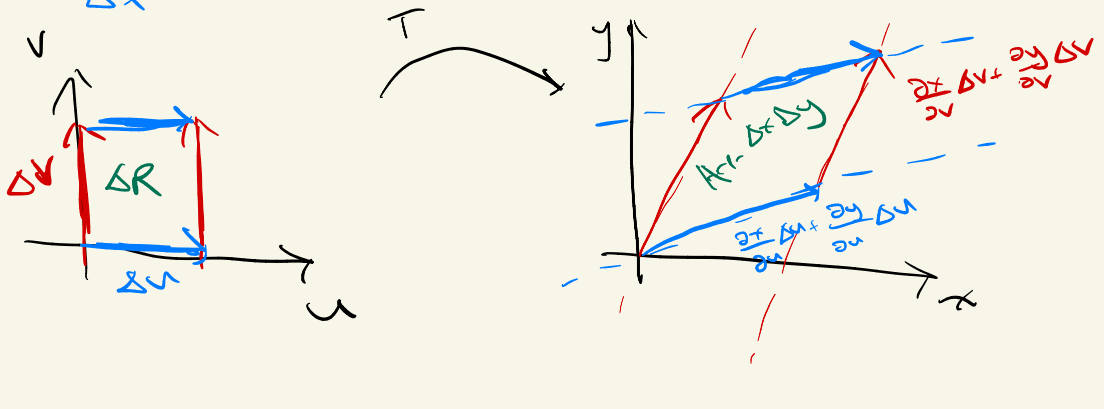
Since

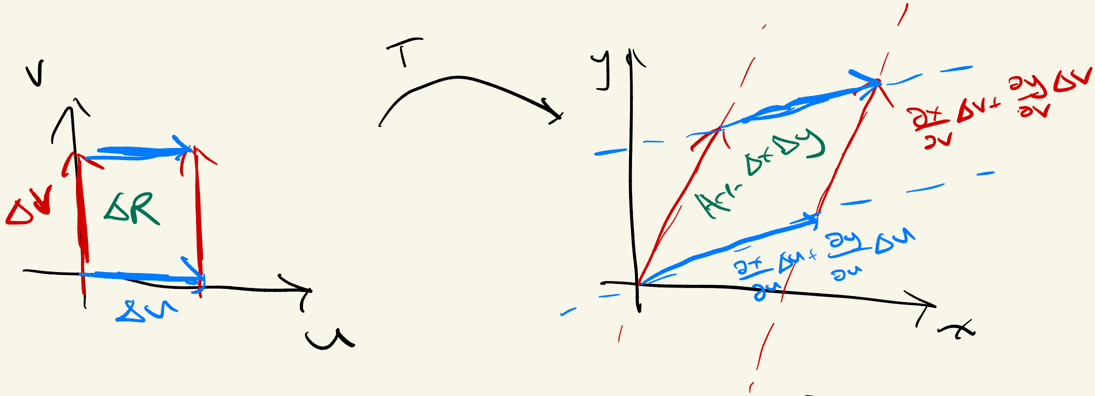
$dx = f'(u)du$

$$\begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial u}\Delta u + \frac{\partial x}{\partial v}\Delta v \\ \frac{\partial y}{\partial u}\Delta u + \frac{\partial y}{\partial v}\Delta v \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix}}_{J(u,v)} \begin{bmatrix} \Delta u \\ \Delta v \end{bmatrix}$$

Total change in Δx = change due to Δu + change due to Δv





What is the area of this new shape?

Area of parallelogram = det(Jacobian)

$$= \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

$$\Rightarrow \quad \Delta R = \det(\text{Jacobian}) \cdot \Delta x \, \Delta y$$

$$\Rightarrow \quad dR = \underbrace{\det(J(u,v))}_{\text{"like a derivative"}} \cdot dx \, dy$$

Chain rule works:

1D: $$\int_a^b F(x)\,dx = \int_{T^{-1}(a)}^{T^{-1}(b)} f(T(u)) \underbrace{T'(u)}_{\text{derivative}} \, du$$

2D: $$\iint_R f(x,y)\,dx\,dy = \iint_{T^{-1}(R)} f(T(x,y)) \det(J(u,v))\,du\,dv$$

(note - higher dims are the same)

How to calculate determinant (for 2x2)?

$$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$$

$-$ $+$

So Jacobian determinant:

$$\det\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

Ex:

Evaluate $\displaystyle\int_0^4 \int_{y/2}^{(y/2)+1} \frac{2x-y}{2}\,dx\,dy$

By applying transformation

$$u = \frac{2x-y}{2} \qquad v = \frac{y}{2}$$

Soln:

Two Jobs: (1) Figure out bound transformations

(2) Figure out Jacobian determinant

$v = \frac{y}{2}$

$u = x - \frac{y}{2}$

Isolate x, y

$y = 2v$

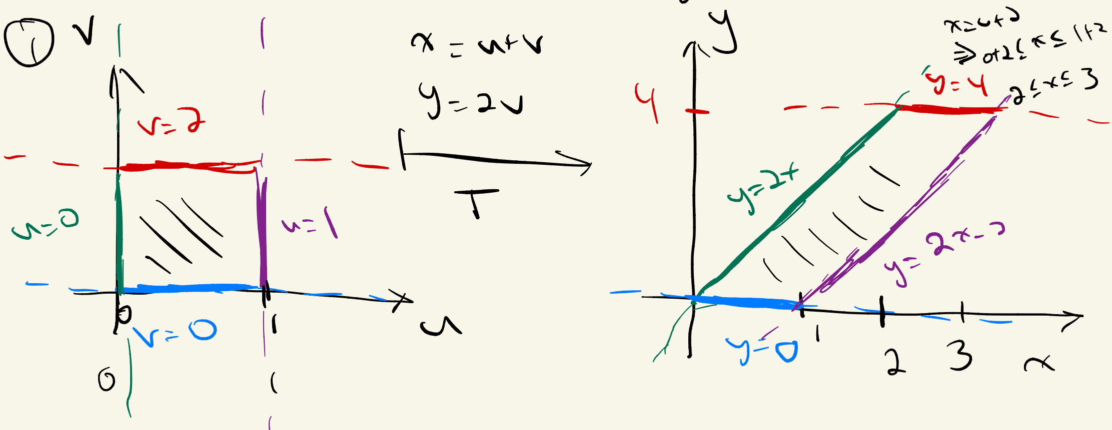
$x = u + \frac{y}{2}$

$= u + v$

$$\boxed{\begin{aligned} x &= u+v \\ y &= 2v \end{aligned}} \qquad (x,y) = T(u,v)$$

Finding boundaries = Tracking equations' transformations

(1)



$$\det(J(u,v)) = \det\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}$$

$$= \det\begin{pmatrix} \frac{\partial}{\partial u}(u+v) & \frac{\partial}{\partial v}(u+v) \\ \frac{\partial}{\partial u}(2v) & \frac{\partial}{\partial v}(2v) \end{pmatrix}$$

$$= \det\begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$$

$$= 2\cdot 1 - 0\cdot 1$$

$$= 2$$

look at region on last page

$$\int_{y=0}^{y=4}\int_{x=\frac{y}{2}}^{x=\frac{y}{2}+1} \frac{2x-y}{2}\,dx\,dy = \int_{v=0}^{v=2}\int_{u=0}^{u=1} \frac{2(u+v)-2v}{2}\det(J(u,v))\,du\,dv$$

$$= \int_0^2\int_0^1 u \cdot 2\,du\,dv$$

$$= \int_0^2 \left[u^2\right]_0^1 dv$$

$$= \int_0^2 1\,dv$$

$$= 2$$

<u>Midterm</u>: open note, <u>not</u> open internet

- 2 parts, split in half
  - 3 questions per part
- All chapter 15 (that we've covered in class).

  Nothing beyond this.

  (up to and including today, 15.7)

15.1 – Riemann sums

15.2 – Integrals over rectangles and general regions

- Fubini
- sketching!
- Bounding!

15.3 – Area and average value (both defs)

15.4 – Polar

- converting, finding limits
- understand how area element works

15.5 – Triple integrals

- limits, sketch

15.6 – Mass, moment of inertia ←

No parallel axis theorem but these formulas are in your notes and these problems are fair game

- 15.7 Cylindrical, Spherical
  - find limits, convert, volume elements

- 15.8 Jacobian
  - If I give you a transformation, you should be able to compute the transformed integral.